The Orbit Equivalence Class of The Non-Singular Shift

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Product measures

Definition (product measures)

A probability measure μ on $2^{\mathbb{Z}}$ is a **product measure** if the coordinates X_n, $n \in \mathbb{Z}$, are independent w.r.t. μ.

- A product measure is denoted $\mu = \bigotimes_{n\in\mathbb{Z}} \mu_n$.
- $\mu = \bigotimes_{n\in\mathbb{Z}} \mu_n$ is **regular** if there is $\delta > 0$ s.t. $\mu_n(a) \geq \delta$.

Theorem (Kakutani's dichotomy (48'))

If $\mu = \bigotimes_{n\in\mathbb{Z}} \mu_n$ and $\nu = \bigotimes_{n\in\mathbb{Z}} \nu_n$ are supported on the same cylinders, they are either equivalent or totally singular.

Moreover: $\mu \sim \nu \iff \sum_{n\in\mathbb{Z}} \sum_a \left(\sqrt{\mu_n(a)} - \sqrt{\nu_n(a)}\right)^2 < \infty.$

Markov measures I

Definition (Markov measures)

A probability measure μ on $2^{\mathbb{Z}}$ is a **Markov measure** if for every n, conditionally on X_n, the mutual distribution of $\ldots, X_{n-2}, X_{n-1}$ is independent of $X_{n+1}, X_{n+2}, \ldots$.

- A Markov measure is denoted $\mu = \mu_{(P_n)_{n\in\mathbb{Z}}}$ where $(P_n)_{n\in\mathbb{Z}}$ is the stochastic matrix $P_n(a,b)=\mu(X_{n+1}=b|X_n=a)$.
- $\mu_{(P_n)_{n\in\mathbb{Z}}}$ is **regular** if there is a positive integer M and $\delta > 0$ s.t. $\mu(X_{n+M}=b|X_n=a)=P_n\cdots P_{n+M}(a,b)\geq\delta$.
- In order for the shift to be non-singular, all the matrices P_n need to share the same support. Thus, the support of a Markov measure is a sub-shift of finite type.

Markov measures II

- Kakutani's dichotomy fails for general Markov measures.
- The reason is that Markov measures do not have tail 0-1 law.

Theorem (A-R (22'))

Regular *Markov measures* $\mu = \mu_{(P_n)_{n\in\mathbb{Z}}}$ *and* $\nu = \nu_{(Q_n)_{n\in\mathbb{Z}}}$ *are either equivalent or totally singular.*
There is a quantitative criteria, generalizing Kakutani's criteria, to decide between these alternatives.

- The proof is purely probabilistic in nature.
- If μ is a Markov measure on $2^{\mathbb{Z}}$, the push-forward of μ by the shift is again a Markov measure.

Recap I

- (X,μ) is a standard non-atomic measure space.
- $\mathrm{Aut}\,(X,[\mu])$ is the group of invertible transformations $T : X \to X$ s.t. μ and $\mu \circ T$ are equivalent.
- For $T \in \mathrm{Aut}\,(X,[\mu])$ the R-N cocycle is $\psi_T(x) = \log \frac{d\mu\circ T}{d\mu}(x)$.
- A **non-singular action** $G \curvearrowright (X,\mu)$ is a group-homomorphism $G \to \mathrm{Aut}\,(X,[\mu])$.

Ergodicity: $G \curvearrowright (X,\mu)$ is **ergodic** if every set $A \subset X$ with $G.A \subset A$ satisfies $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

Recurrence: $G \curvearrowright (X,\mu)$ is **recurrent** if for every set $A \subset X$ there is a $g \in G$ s.t. $\mu(A \cap g.A) > 0$.

Recap II

- $G \curvearrowright (X, \mu)$ is a non-singular action.
- $r \in \mathbb{R}$ is an **essential value** if for $A \subset X$ and $\epsilon > 0$, there is a scheme $g : B \to g(B)$ in A s.t. $|\psi_g(x) - r| < \epsilon$ for $x \in B$.
- The **ratio set** $\mathrm{RatioSet}(G, \mu) \subset \mathbb{R}$ is the closed subgroup of all essential values.
- Then the ratio set is either of the following: $\{0\}$ (Type III_0), $\mathbb{Z} \log \lambda$ (Type III_λ), $\mathbb{R}$ (Type III_1).
- For amenable groups, the ratio set is a complete invariant of o.e. (outside the notorious Type III_0).

The orbit equivalence class of the shift II

Theorem (Bjorklund, Kosloff & Vaes (21'); A-R (22'))

The only possible o.e. classes for regular Markov shift are Type II *and Type* III_1.

Moreover: *Type* II *occurs iff* μ *is equivalent to a stationary Markov measure* ($P_n = P$ *for all* n).

- The exact classification of the shift was completely unknown until the last decade.
- Before that, only specific constructions of Bernoulli shifts of Type III were known, but not the exact Type III_λ.

Corollary

The only way in which a regular Markov shift is probability preserving, is when it is a classical stationary Markov chain.

The orbit equivalence class of the shift III

- If $\mu = \mu_{(P_n)_{n\in\mathbb{Z}}}$ is a Markov shift,

$$\psi_n(x)=\log \frac{d\mu\circ T^n}{d\mu}(X)=\log \prod_{k\in\mathbb{Z}} \frac{P_{k-n}(x_k,x_{k+1})}{P_k(x_k,x_{k+1})}.$$

- Outside certain examples, this is extremely complicated.

Theorem (A-R (22'))

If μ is a regular Markov shift, $\mathrm{RatioSet}\,(\Pi,\mu) \subset \mathrm{RatioSet}\,(T,\mu)$.

- In this formulation, "Π" stands for the non-singular modification of the finite permutations.

The asymptotic group I

Definition

Let $G \curvearrowright (X,\mu)$ be a non-singular action. Fix a compatible metric d on X. The **asymptotic group** to G is the class G' of all $T \in \mathrm{Aut}\,(X,[\mu])$ s.t.

- d-asymptotic: $d(g.x, g.Tx) \xrightarrow[g\to\infty]{} 0$ for almost every $x \in X$.
- μ-asymptotic: $\left|\psi_g(x) - \psi_{g\circ T}(x)\right| \xrightarrow[g\to\infty]{} 0$ for almost every $x \in X$.

- Using the triangle inequality (for d-asymptotic) and the cocycle property (for μ-asymptotic), the asymptotic group is indeed a group.
- In general, no containment relations between G and G'.

The asymptotic group II

Example ("E_0-actions are asymptotic to $E(G,2)$-actions")

$X = 2^{\mathbb{Z}}$ with the metric $d(x,y) = 2^{-\inf\{|n|:x_n \neq y_n\}}$ and $G = \mathbb{Z}$ acting by shift.

- It is easy to see that Π is d-asymptotic to T.
- If μ is a regular Bernoulli shift, Π is also μ-asymptotic to T.
- If μ is a regular Markov shift this is no longer true.
- However, there are enough elements of Π that are μ-asymptotic to the shift.
- Thus, $\Pi \cap G'$ is a rich group.

The asymptotic group III

Theorem (A-R (22'))

Let $G \curvearrowright (X,\mu)$ be a non-singular recurrent action of amenable group. For $f : X \to \mathbb{C}$, if f is G-invariant then it is G'-invariant.

- *In particular, if G' is ergodic then G is ergodic.*

- The ratio set can be characterized using the invariant functions of the "Maharam extension".
- Then $e(G',\mu) \subset e(G,\mu)$ is a natural generalization of the above theorem.

The asymptotic group IV (on the proof)

Proof.

For simplicity assume $G=\mathbb{Z}$ acts via $T\in\mathrm{Aut}(X,\mu)$ (measure preserving) and S is asymptotic to T.

- For $f:X\to\mathbb{R}$ uniformly continuous, $f(T^n x)-f(T^n(Sx))\xrightarrow[|n|\to\infty]{}0$.
- By the ergodic theorem

$$\mathbb{E}(f|\mathrm{Inv}(T))\circ S=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^n(Sx))$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^n x)=\mathbb{E}(f|\mathrm{Inv}(T)).$$

- Approximating in L^1, the same holds for integrable f.
- If $f\circ T=f$ then $f\circ S=\mathbb{E}(f|\mathrm{Inv}(T))\circ S=\mathbb{E}(f|\mathrm{Inv}(T))=f$.

□

The asymptotic group V (on the proof)

- Same argument works for non-singular transformations using:

Theorem (Hurewicz's ergodic theorem)

Let $T \in \mathrm{Aut}(X,[\mu])$ *and set* $w_n(x) = \frac{d\mu \circ T^n}{d\mu}(x)$. *Then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} w_n(x) f(T^n x)}{\sum_{n=1}^{N} w_n(x)} \text{ exists for almost every } x \in X.$$

- For prob. preserving actions of amenable groups the ergodic theorem holds along Følner sequences (Lindenstrauss).
- However, for non-singular actions of general groups (even Abelian) the ergodic theorem fails.

The asymptotic group VI (on the proof)

- Typically, an "ergodic theorem" for an action $G \curvearrowright (X, \mu)$ is a statement of the following form:

$$\exists F_n \nearrow G, \forall f, \text{ the ergodic averages of } f \text{ along } (F_n) \text{ converge.}$$

- However, for the sake of the above argument one may relax this to the following weaker assertion:

$$\forall f, \exists F_n \nearrow G, \text{ the ergodic averages of } f \text{ along } (F_n) \text{ converge.}$$

- We call this weaker version **ad hoc ergodic theorem**.

Theorem (Danilenko (19'))

Every non-singular free action of an amenable group satisfies the ad hoc ergodic theorem.

The asymptotic group VII (on the proof)

- The proof of the ad hoc ergodic theorem relies on:
    1. μ-hyperfiniteness of actions of amenable groups (Connes, Feldman & Weiss + Slaman & Steel).
    2. The martingale convergence theorem: if $\mathcal{F}_n \nearrow \mathcal{F}$ are σ-algebras then $\mathbb{E}(\cdot \mid \mathcal{F}_n) \to \mathbb{E}(\cdot \mid \mathcal{F})$.
- This reasoning goes beyond amenable groups, since non-amenable groups admit hyper-finite actions.

The asymptotic group VII (on the proof)

Example

Let F_r be the free group of rank $r \geq 2$ and $F_r \curvearrowright (\partial F_r, \nu)$.

- ∂F_r is a sub-shift of finite type of $\{s_1, \ldots, s_r\}^\omega$ and ν is a natural Markov measure.
- It is non-singular, recurrent, ergodic and Type III.
- Its orbit equivalence relation E_t is hyper-finite (Connes, Feldman & Weiss; Dougherty, Jackson & Kechris).

- This is true for every non-elementary hyperbolic group acting on its Gromov boundary with a Patterson-Sullivan measure.
- These actions are hyper-finite (Adams; Marquis & Sabok).

Thank you